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A Study on the Differential Problem

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Abstract

This study uses the mathematical software Maple for the auxiliary tool to study the differential problem of two types of functions. We can obtain the closed forms of any order derivatives of these two types of functions by using Leibniz differential rule. In addition, we provide two examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. Therefore, Maple provides insights and guidance regarding problem-solving methods.

Keywords: derivatives; closed forms; Leibniz differential rule; Maple.

1. INTRODUCTION

As information technology advances, whether computers can become comparable with human brains to perform abstract tasks, such as abstract art similar to the paintings of Picasso and musical compositions similar to those of Beethoven, is a natural question. Currently, this appears unattainable. In addition, whether computers can solve abstract and difficult mathematical problems and develop abstract mathematical theories such as those of mathematicians also appears unfeasible. Nevertheless, in seeking for alternatives, we can study what assistance mathematical software can provide. This study introduces how to conduct mathematical research using the mathematical software Maple. The main reasons of using Maple in this study are its simple instructions and ease of use, which enable beginners to learn the operating techniques in a short period. By employing the powerful computing capabilities of Maple, difficult problems can be easily solved. Even when Maple cannot determine the solution, problem-solving hints can be identified and inferred from the approximate values calculated and solutions to similar problems, as determined by Maple. For this reason, Maple can provide insights into scientific research. Inquiring through an online support system provided by Maple or browsing the Maple website (www.maplesoft.com) can facilitate further understanding of Maple and might provide unexpected insights. For the instructions and operations of Maple, we can refer to (Robertson, 1996; Garvan, 2001; Richards, 2002; Dodson & Gonzalez, 1995; Abell & Braselton, 2005; Stroeker & Kaashoek, 1999; Tocci & Adams, 1996).

In calculus and engineering mathematics curricula, finding $f^{(n)}(c)$ (the n-th order derivative value of function $f(x)$ at $x=c$), in general, necessary goes through two procedures: Firstly evaluating $f^{(n)}(x)$ (the n-th order derivative of $f(x)$), and secondly substituting $x=c$ to $f^{(n)}(x)$. When evaluating the higher order derivative values of a function (i.e. n is large), these two procedures will make us face with increasingly complex calculations. Therefore, to obtain the answers through manual calculations is not an easy thing. In this paper, we mainly evaluate the derivatives of the following two types of functions

$$f(x)=x^{\lambda}e^{ax}\cos(bx+c) \tag{1}$$

$$g(x)=x^{\lambda}e^{ax}\sin(bx+c) \tag{2}$$

Where λ, a, b, c are real numbers. We can obtain the closed forms of any order derivatives of these two types of functions by using Leibniz differential rule; these are the main results of this paper (i.e., Theorems 1, 2). As for the study of related differential problems can refer to (Yu, (a)–(h)). On the other hand, we provide two examples to do calculation practically. The research methods adopted in this study involved finding solutions through manual calculations and verifying these solutions by using Maple. This type of research method not only allows the discovery of calculation errors, but also helps modify the original directions of thinking from manual and Maple calculations. For this reason, Maple provides insights and guidance regarding problem-solving methods.

2. Main Results

Firstly, we introduce a notation and a formula used in this study.

Notation.

Suppose t is any real number, and m is any positive integer. Define $(t)_m = t(t-1)\cdots(t-m+1)$, and $(t)_0 = 1$.

Leibniz differential rule (Apostol , 1975, p121).

Suppose n is an non-negative integer, $f(x)$ and $g(x)$ are n-times differentiable functions. Then the n-th order derivative of the product function $f(x)\cdot g(x)$,

$$(f\cdot g)^{(n)} = \sum_{k=0}^{n}\binom{n}{k} f^{(n-k)} g^{(k)}$$

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The following is the first major result of this study, we determine the the closed forms of any order derivatives of function (1).

Theorem 1. Assume λ, a, b, c are real numbers, n is any positive integer. Suppose the domain of the function $f(x) = x^{\lambda} e^{ax} \cos(bx + c)$ is $\{x \in R | x^{\lambda} exist, x \neq 0\}$. Then the n-th order derivative of $f(x)$,

$$f^{(n)}(x) = e^{ax} \cdot \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k}\binom{k}{m} (\lambda)_{n-k} a^{k-m} b^m x^{\lambda-n+k} \cos\left(bx + c + \frac{m\pi}{2}\right) \quad (3)$$

for all x satisfy x^{λ} exist and $x \neq 0$.

Proof. $f^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} (x^{\lambda})^{(n-k)} [e^{ax} \cos(bx + c)]^{(k)}$ (by Leibniz differential rule)

$$= \sum_{k=0}^{n} \binom{n}{k} (\lambda)_{n-k} x^{\lambda-n+k} \sum_{m=0}^{k} \binom{k}{m} (e^{ax})^{(k-m)} [\cos(bx + c)]^{(m)}$$

(again by Leibniz differential rule)

$$= e^{ax} \cdot \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k}\binom{k}{m} (\lambda)_{n-k} a^{k-m} b^m x^{\lambda-n+k} \cos\left(bx + c + \frac{m\pi}{2}\right)$$ ■

Next, we obtain the the closed forms of any order derivatives of function (2).

Theorem 2. The same assumptions are the same as Theorem 1. Suppose the domain of the function $g(x) = x^{\lambda} e^{ax} \sin(bx + c)$ is $\{x \in R | x^{\lambda} exist, x \neq 0\}$. Then the n-th order derivative of $g(x)$,

$$g^{(n)}(x) = e^{ax} \cdot \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k}\binom{k}{m} (\lambda)_{n-k} a^{k-m} b^m x^{\lambda-n+k} \sin\left(bx + c + \frac{m\pi}{2}\right) \quad (4)$$

for all x satisfy x^{λ} exist and $x \neq 0$.

Proof. $g^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} (x^{\lambda})^{(n-k)} [e^{ax} \sin(bx + c)]^{(k)}$ (by Leibniz differential rule)

$$= \sum_{k=0}^{n} \binom{n}{k} (\lambda)_{n-k} x^{\lambda-n+k} \sum_{m=0}^{k} \binom{k}{m} (e^{ax})^{(k-m)} [\sin(bx + c)]^{(m)}$$

(again by Leibniz differential rule)

$$= e^{ax} \cdot \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \binom{k}{m} (\lambda)_{n-k} a^{k-m} b^m x^{\lambda-n+k} \sin\left(bx + c + \frac{m\pi}{2}\right) \qquad \blacksquare$$

3. Examples

In the following, we provide two functions to determine the closed forms of their any order derivatives and evaluate some higher order derivative values practically. On the other hand, we use Maple to calculate the approximations of these higher order derivative values and their closed forms for verifying our answers.

Example 1. Suppose the domain of the function $f(x) = x^{3/4} e^{2x} \cos\left(5x - \frac{2\pi}{3}\right)$ is $\{x \in R | x > 0\}$ (the case of $\lambda = 3/4, a = 2, b = 5, c = -2\pi/3$ in Theorem 1). Then the n -th order derivative of $f(x)$,

$$f^{(n)}(x) = e^{2x} \cdot \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \binom{k}{m} (3/4)_{n-k} \cdot 2^{k-m} \cdot 5^m \cdot x^{3/4-n+k} \cos\left(5x - \frac{2\pi}{3} + \frac{m\pi}{2}\right) \qquad (5)$$

for all $x > 0$.

Therefore, we can determine 11-th order derivative of $f(x)$ at $x = 3$,

$$f^{(11)}(3) = e^6 \cdot \sum_{k=0}^{11} \sum_{m=0}^{k} \binom{11}{k} \binom{k}{m} (3/4)_{11-k} \cdot 2^{k-m} \cdot 5^m \cdot 3^{-41/4+k} \cos\left(15 - \frac{2\pi}{3} + \frac{m\pi}{2}\right) \qquad (6)$$

Next, we use Maple to verify the correctness of (6).

```
>f:=x->x^(3/4)*exp(2*x)*cos(5*x-2*Pi/3);
```

$$f := x \to x^{3/4} e^{2x} \cos\left(5x - \frac{2}{3}\pi\right)$$

```
>evalf((D@@11)(f)(3),14);
```

$$1.1410354758423 \cdot 10^{11}$$

```
>evalf(exp(6)*sum(sum(11!/(k!*(11-k)!)*k!/(m!*(k-m)!)*product(3/4-j,j=0..(10-k))*2^(k-m)*5^m*3^(-41/4+k)
*cos(15-2*Pi/3+m*Pi/2),m=0..k),k=0..11),14);
```

$$1.1410354758423 \cdot 10^{11}$$

Example 2. Assume the domain of the function $g(x) = x^{11/3}e^{-3x}\sin\left(8x + \frac{3\pi}{4}\right)$ is $\{x \in R \mid x \neq 0\}$ (the case of $\lambda = 11/3, a = -3, b = 8, c = 3\pi/4$ in Theorem 2). Then the n -th order derivative of $g(x)$,

$$g^{(n)}(x) = e^{-3x} \cdot \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\binom{k}{m}(11/3)_{n-k} \cdot (-3)^{k-m} \cdot 8^m \cdot x^{11/3-n+k} \sin\left(8x + \frac{3\pi}{4} + \frac{m\pi}{2}\right) \tag{7}$$

for all $x \neq 0$.

Thus, we can obtain 13-th order derivative of $g(x)$ at $x = 4$,

$$g^{(13)}(4) = e^{-12} \cdot \sum_{k=0}^{13}\sum_{m=0}^{k}\binom{13}{k}\binom{k}{m}(11/3)_{13-k} \cdot (-3)^{k-m} \cdot 8^m \cdot 4^{-28/3+k} \sin\left(32 + \frac{3\pi}{4} + \frac{m\pi}{2}\right) \tag{8}$$

We also use Maple to verify the correctness of (8).

>g:=x->x^(11/3)*exp(-3*x)*sin(8*x+3*Pi/4);

$$g := x \to x^{11/3}\, e^{-3x} \sin\left(8x + \frac{3}{4}\pi\right)$$

>evalf((D@@13)(g)(4),14);

$$1.0651020025558 \cdot 10^9$$

>evalf(exp(-12)*sum(sum(13!/(k!*(13-k)!)*k!/(m!*(k-m)!)*product(11/3-j,j=0..(12-k))*(-3)^(k-m)*8^m*4^(-28/3+k)*sin(32+3*Pi/4+m*Pi/2),m=0..k),k=0..13),14);

$$1.0651020025558 \cdot 10^9$$

4. Conclusion

From the above discussion, we know the Leibniz differential rule plays a significant role in the theoretical inferences of this study. In fact, the application of this formula is extensive, and can be used to easily solve many difficult problems; we endeavor to conduct further studies on related applications. On the other hand, Maple also plays a vital assistive role in problem-solving. In the future, we will extend the research topic to other calculus and engineering mathematics problems and solve these problems by using Maple. These results will be used as teaching materials for Maple on education and research to enhance the connotations of calculus and engineering mathematics.

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