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GENERALISED GAUSSIAN QUADRATURE OVER A CIRCLE

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Abstract

This paper presents a Generalised Gaussian quadrature method for numerical integration over a circle bounded by the region $\left\{(x,\ y)\ /\ 0\ \leq\ x \leq\ a,\ 0 \leq\ y\ \leq\ \sqrt{a^2 - x^2}\right\}$, are derived using transformation of variables, a general formula for integration over the above-mentioned region is provided. New Gaussian points and corresponding weights are calculated. The numerical evaluation of circle domain integrals of any arbitrary functions is illustrated with some numerical examples.

Keywords: Finite element method; Quadrature rule; circle region; extended numerical integration.

1. INTRODUCTION

The finite element method is essentially a numerical method for the approximate solution of practical problems arising in engineering and scientific analysis. The advantages of the finite element technique over other alternatives are more fully appreciated in two dimensional situations. Surface integrals are used in multiple areas of physics and engineering. In particular, they are used for Problems involving calculations of mass of a shell, center of mass and moments of inertia of a shell, fluid flow and mass flow across a surface, electric charge distributed over a surface, plate bending, plane strain, heat conduction over a plate, and similar problems in other areas of engineering which are very difficult to analyse using analytical techniques, These problems can be solved using the finite element method.

From the literature review we may realize that several works in numerical integration using Gaussian quadrature over triangle region and linear convex quadrilateral region have been carried out [1-6,12]. The both Gaussian and Szego quadrature formulae depend on the location of the singularities of the integrand f(z) with respect to the unit circle given in [8-9]

The method proposed here is termed as Generalized Gaussian rules, since the Generalized Gaussian quadrature nodes and weights for products of polynomial and logarithmic function given in [13] by Ma et al. are used in this paper

The paper is organized as follows. In Section 2 we will introduce the Generalized Gaussian quadrature formula over a circle region of various values of radius a. and In Section 3 we compare the numerical results with some illustrative examples.

2. Generalized Gaussian quadrature over the region C

Generalized Gaussian quadrature rule for integrating function bounded by the region $C = \left\{(x,\ y)\ /\ 0 \le\ x \le\ a\ ,\ 0 \le\ y\ \le\ \sqrt{a^2 - x^2}\right\}$, Few such regions are plotted below

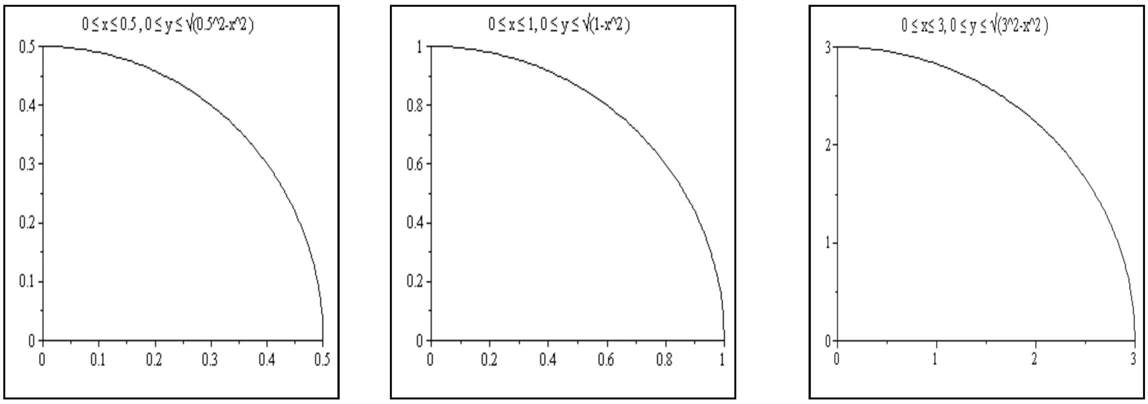

Figure.1: Region C_1 , C_2 and C_3

Any such region can be represented as C_1 , C_2 and C_3 where

$C_1 = \left\{(x, y)\ /\ 0\ \le\ x \le\ 0.5, 0\ \le\ y\ \le\ \sqrt{0.5^2 - x^2}\right\}$,

$C_2 = \left\{(x, y)\ /\ 0\ \le\ x \le\ 1, 0\ \le\ y\ \le\ \sqrt{1^2 - x^2}\right\}$ and

$C_3 = \left\{(x, y)\ /\ 0\ \le\ x \le\ 3, 0\ \le\ y\ \le\ \sqrt{3^2 - x^2}\right\}$

2.1 Formation of integrals over a circle

The Numerical integration of an arbitrary function f over a circle is given by

$$I = \iint_C f(x,y)dx\,dy = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)dy\,dx = 4\int_0^a\int_0^{\sqrt{a^2-x^2}} f(x,y)dy\,dx \qquad (1)$$

Where a is the radius of the circle

The integral of the eqn. (1) can be transformed to the square $\{(\ \xi\ ,\ \eta\ )\ /\ 0 \le\ \xi \le 1\ ,\ 0 \le \eta \le 1\}$. Transformation is

$$x =\ a\,\xi \ \text{ and } y = a\,\eta\,\sqrt{1-\xi^2} \qquad (2)$$

We have

$$I=\int_0^a\int_0^{\sqrt{a^2-x^2}} f(x,y)dy\,dx=\int_0^1\int_0^1 f\big(x(\xi\,,\eta), y(\xi\,,\eta)\big)\,J\,d\xi\,d\eta \qquad (3)$$

Where $J\,(\xi\,,\eta)$ is the Jacobians of the transformation

$$J(\xi\,,\eta) = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix} = a^2\sqrt{1-\xi^2}$$

From eqn.(3) , we can write as

$$I = \int_0^1 \int_0^1 f\left(a\,\xi\,, a\,\eta\,\sqrt{1-\xi^2}\right) a^2\sqrt{1-\xi^2}\,d\xi\,d\eta$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a^2\sqrt{1-\xi^2}\;\; w_i w_j\, f(x(\xi_i\,,\eta_j)\,, y(\xi_i\,,\eta_j)) \qquad (4)$$

Where $\xi_{i\,,}\,\eta_j$ are Gaussian points and w_i, w_j are corresponding weights. We can rewrite eqn. (4) as

$$\mathrm{I} = \sum_{k}^{N=n\times n} W_k f\,(x_k, y_k) \qquad (5)$$

Where $W_k = a^2\sqrt{1-\xi^2}\;\; w_i w_j$, (5a)

$x_k = a\,\xi$, (5b)

$y_k = a\,\eta\,\sqrt{1-\xi^2}$, (5c)

if $k = 1,2,3,----$, $i,j = 1,2,3,----$

We find out new Gaussian points x_k , y_k and weights coefficients W_k of various order N =5,10,15,20 by using eqn. (5a), (5b) and (5c) and Tabulated in Table 1

Table. 1 Gaussian points and weights coefficient over the region C_1 for N = 5

x_k	y_k	W_k
0.002826114	0.002826069	0.000110742
0.036715186	0.002818485	0.000685881
0.142478702	0.002708944	0.001461138
0.309741132	0.00221853	0.001446592
0.457879042	0.001135335	0.000440357
0.002826114	0.036714599	0.000687727
0.036715186	0.036616068	0.004259454
0.142478702	0.035192981	0.009073948
0.309741132	0.028821812	0.008983611
0.457879042	0.014749598	0.002734699
0.002826114	0.142476426	0.001524313
0.036715186	0.142094059	0.009440868
0.142478702	0.136571561	0.020111952
0.309741132	0.111847299	0.019911726
0.457879042	0.057237994	0.006061324
0.002826114	0.309736184	0.001842738
0.036715186	0.308904938	0.011413041
0.142478702	0.296899321	0.024313287
0.309741132	0.243150088	0.024071235
0.457879042	0.124432359	0.007327519
0.002826114	0.457871727	0.001096133
0.036715186	0.456642927	0.006788925
0.142478702	0.438895459	0.014462499
0.309741132	0.359439925	0.014318516
0.457879042	0.183943827	0.004358696

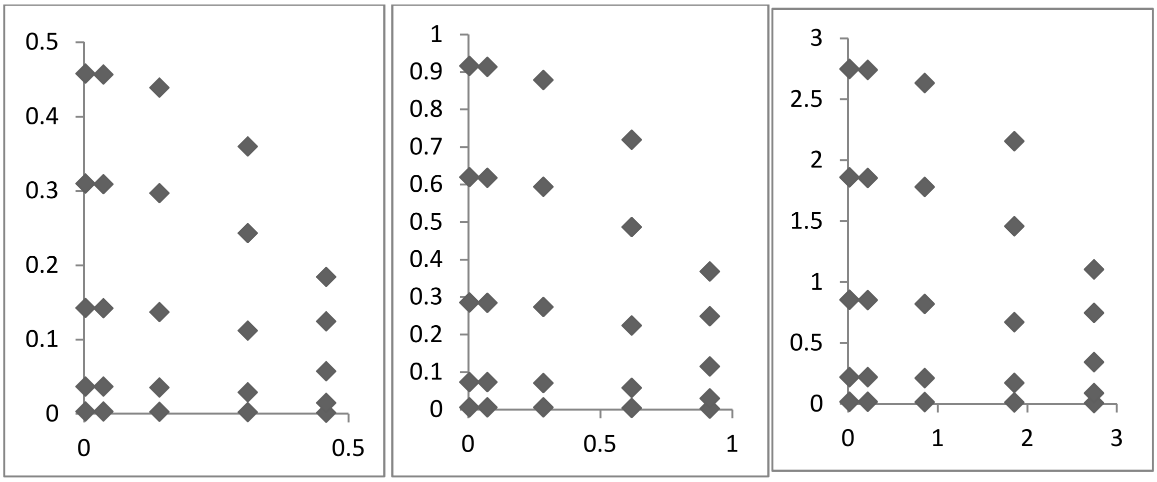

Figure. 2 (x_k, y_k) values for N = 5 in the region C_1, C_2 and C_3

3. Numerical results

Exact value	Order	Computed value
1) $\int_0^{0.5}\int_0^{\sqrt{0.5^2-x^2}} e^{-y^2}\sin(x+y)\,dydx = 0.07433757139$	N=5 N=10 N=15 N=20	0.074633535566509 0.074768903251688 0.074336920378733 0.074343147813341
2) $\int_0^{0.5}\int_0^{\sqrt{0.5^2-x^2}} \sqrt{x+y}\log(\frac{1}{x+y})\,dydx = 0.1065939432$	N=5 N=10 N=15 N=20	0.106862527487902 0.105948446063608 0.106586696412228 0.106599618805168
3) $\int_0^{0.5}\int_0^{\sqrt{0.5^2-x^2}} \frac{x^4+y^4}{1+x^2y}\,dydx =$ 0.003021635575	N=5 N=10 N=15 N=20	0.003068473535678 0.003044894025658 0.003023337422748 0.003022371527804
4) $\int_0^{1}\int_0^{\sqrt{1^2-x^2}} \frac{1}{\sqrt{1+x+y}}\,dydx = 0.5845377364$	N=5 N=10 N=15 N=20	0.586143927526821 0.584013129291003 0.584657843965136 0.584530430190500
5) $\int_0^{3}\int_0^{\sqrt{3^2-y^2}} \frac{xy}{\sqrt{x^2+y^2}}\,dxdy = 4.500000000$	N=5 N=10 N=15 N=20	4.500436035779733 4.546595533503721 4.499437632134709 4.499999951395337
6) $\int_0^{1}\int_0^{\sqrt{1^2-x^2}} \frac{e^{-x}}{\sqrt{1-xy}}\,dydx = 0.5787385144$	N=5 N=10 N=15 N=20	0.579547828268574 0.575803194275310 0.578875488467790 0.578755463118211

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7) $\int_0^3 \int_0^{\sqrt{3^2-x^2}} \sqrt{x+y}dydx = 11.03825541$	N=5 N=10 N=15 N=20	11.07538413171581 11.07645062614108 11.03855822062576 11.03898023226206

4. Conclusions

In this paper we derived Generalised Gaussian quadrature method for calculating integral over a circle region

$\{(x, y) \ / \ 0 \leq x \leq a, 0 \leq y \leq \sqrt{a^2 - x^2}\}$ with radius a = 0.5, 1, 3. new Gaussian points and its weights are calculated of various order N = 5, 10, 15, 20. We have then evaluate the typical integrals Governed by the proposed method. The results obtained are in excellent agreement with the exact value.

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